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Chapter 6

ORIGIN OF THE BASIC EQUATIONS OF ROCKET DYNAMICS*

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For scientific exactness the history of the development of individual scientific ideas and theories and the origin of particular relations are essential for a better understanding of the science itself and for further successful work related to these theories. Therefore, it is of interest to determine how the basic equations for the mechanics of the body of variable mass have been derived, as without any doubt, they are the basis of modern rocket dynamics. By the body of variable mass is meant, as we know, such a body whose mass changes by simple mechanical attaching or detaching of parts and not in a relativistic sense, dependent on velocity. The mass of many natural bodies changes in this sense mechanically. The mass of the Earth changes as a result of the fall of meteorites, and the mass of the Sun as a result of cosmic dust and radiation. The mass of the meteorites themselves changes during the flight through the atmosphere due to separating of parts and combustion. On the other hand, there are many bodies whose mass is artificially and intentionally changed: by serostats when throwing away the ballast, and by rockets when simply ejecting parts, or by exhausting the combustion gases.

Consequently, the problem of motion of such, either natural or artificial bodies, where the mechanism of increasing, resp. decreasing, the mass has been determined, exists and is, no doubt, very important. However, the classical vector equation of motion, resp. scalar equations, deduced from the second law of Newton are not valid, as they assume that *the mass does not change during the motion*, which is now not the case. The equations describing such motion of bodies with mechanically variable masses have been derived, and they are nowadays widely used as introductory and basic *equations of rocket dynamics*, but it is interesting to note that generally there are no indications concerning the origin of these equations. There exist only certain critical considerations in connection with different approaches to their derivation, and that is all. Thus the exact origin of setting up (first deriving) these equations shall by analyzed critically from the standpoint of contents, starting hypotheses and applied methods.

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The first to raise these questions and derive necessary basic equations of such motion was, without any doubt, the Russian scientist Ivan Vsevolodovich Meshchersky (1859-1935). His works, related to this, are: *Dynamics of a point with variable mass* (M. Sc. thesis at the University of Petrograd, published in 1897), and *Equations of the motion of a point with a variable mass in the general case*. Nowadays, when all the variations of deriving basic equations of rocket dynamics have been critically analyzed, it is necessary to point out how they were derived by Meshchersky. The basic hypothesis which he started from was the hypothesis of the contact of expelled and remaining masses. He most likely assumed that if the expelled part of the body with mass $\Delta m > 0$ and the rest of the body with mass $m - \Delta m$ are regarded as two non-elastic solid (or rigid) bodies at the impact, then it is known from mechanical laws that the changes of their respective quantities of motion in the moment of separation (after the impact) are opposite. Accordingly, written in modern vector notation, this is

$$(m - \Delta m)\Delta \vec{v}_r = -(\vec{u} - \vec{v})\Delta m = -\vec{c}\Delta m \,. \qquad (1)$$

Here, $\vec{v}$ is absolute velocity (i.e. velocity in relation to some system of fixed coordinate axes) of the body of mass m before the impact, $\Delta \vec{v}_r$ is the change of this velocity at impact, $\vec{u}$ absolute velocity of the expelled mass Δm after impact and, consequently, $\vec{c} = \vec{u} - \vec{v}$ is the change of velocity of the expelled part in relation to that before impact.

From eq. (1) is obtained at once, simply by neglecting, due to the small magnitude of the second order $\Delta m \, \Delta v_r$, the relation

$$m\Delta \vec{v}_r = -\vec{c}\Delta m,$$

from which, dividing it by Δt and passing to the limit $\Delta t \to 0$, is obtained the expression for the *reactive force* $\vec{F}_r$ (one of basic relations of rocket dynamics)

$$\vec{F}_r = m\frac{d\vec{v}_r}{dt} = -(\vec{u} - \vec{v})\frac{dm}{dt} = -\vec{c}\,\frac{dm}{dt} \,. \qquad (2)$$

Consequently, for the motion of the body of variable mass, on which act the external force $\vec{F}$ and the reactive force $\vec{F}_r$ the *equation of Meshchersky* is derived:

$$m\frac{d\vec{v}}{dt} = \vec{F} + \vec{F}_r = \vec{F} + (\vec{u} - \vec{v})\frac{dm}{dt} \,, \qquad (3)$$

where now $dm/dt < 0$ is introduced as *consumption of mass* in contrast with the relation (2), where $dm/dt > 0$; where dv/dt is the total change owing to the action of the external force and the reactive force.

If it is assumed that the absolute (in the previous sense) velocities $\vec{u}$ of the expelled parts are equal to zero ($\vec{u} = 0$), then from the equation of Meshchersky (3) is obtained the equation

$$\frac{d}{dt}(m\vec{v}) = \vec{F}. \quad (4)$$

In critical retrospect on the way of deriving the basic equation of rocket dynamics by Meshchersky, one must point out, that Meshchersky did not derive his equation in vector form, as done above. He worked in scalar form from the system of the following differential equations

$$\begin{aligned} M\frac{d^2x}{dt^2} &= X + \frac{dM}{dt}(\alpha - \dot{x}), \\ M\frac{d^2y}{dt^2} &= Y + \frac{dM}{dt}(\beta - \dot{y}), \\ M\frac{d^2z}{dt^2} &= Z + \frac{dM}{dt}(\gamma - \dot{z}), \end{aligned} \quad (5)$$

where M is now the mass of the body, x, y, z the coordinates of its center of mass in relation to some system of fixed coordinate axes, $\dot{x}$, $\dot{y}$, $\dot{z}$ respective coordinates of velocity, dM/dt consumption of mass and, finally α, β, γ coordinates of the absolute velocity of the expelled part. These scalar equations are obviously equivalent to the vector equation (3), when taking into consideration the differences in designations.

Consequently, there is no doubt that Meshchersky derived the basic equations of rocket dynamics and they can be found on pages 74 and 231 of the publication I. V. Meshchersky: *Work on Mechanics of Bodies with Variable Masses* (in Russian), in which appear both his works quoted above. It is not necessary to repeat this scalar derivation itself, for it differs from the vector one only in the usual way.

Concerning the initial assumption of the existence of a steady contact between the primary mass and the expelled mass, it would mean that it can only be spoken of a reactive form in this sense when the primary body and the expelled mass were previously in a solid contact. As is known, this is not indispensable, as the reactive force is created also by the exhaustion of liquid and gaseous masses.

Furthermore Meshchersky had in mind in his publication a pre-determined task - the vertical ascent of an aerostat by throwing away the ballast and, consequently, he mainly *dwelt on the separation of the mass.*

Though the origin of basic equations of rocket dynamics clearly results from the previous overview, one is struck by the fact that in the professional literature, the name of Meshchersky is not mentioned or equation (4) is sometimes ascribed to the Italian scientist Tullio Levi-Civita (1873-1941).

Namely, in addition to the quoted works by Meshchersky dealing with the mechanics of the body of variable mass, there exist also the works of Tullio Levi-Civita dealing with the same subject. His works are:

1. Sul moto di un corpo di massa variable. Rend. Accad. dei Lincei, 1928 (6), 8; and
2. Aggiunta a la nota precedente; in the same volume as above; and
3. Ancore sul moto di un corpo di massa variable. Rend. Accad. dei Lincei, 1930.

Consequently, though the basic equations of the motion of a body of variable mass existed, Levi-Civita derived them again, 31 years later, without mentioning Meshchersky. Considering the fact that Levi-Civita was a great scientist and that one cannot reproach him with the incorrectness of having committed plagiarism, this can be explained only by his not knowing about Russian works. This is due to the fact that:

1. They were published in Russian only, and at that time this language was not so widely known in science as it is today, and
2. The motion of bodies of variable mass (rockets) became a subject of wider scientific interest only after the First World War.

Consequently, Levi-Civita rediscovered the basic equations of rocket dynamics independently of Meshchersky. At the present time these equations are correctly named after Meshchersky only in Russian literature and by those close to the Russian science, whereas in Western literature usually no name is mentioned with these equations. The reason for that is maybe the lack of knowledge, and on the other hand, perhaps the wish not to harm the memory of Levi-Civita, who, to judge by all the known facts, derived *bona fide* the same equations after Meshchersky. There is also another thing that could have played a part in this omission of the author's name and that is the perfect elementariness of the derivation of these relations. The conviction could be that it goes almost without saying as it is an entirely natural consequence of the well-known laws of mechanics and that the derivation itself can be performed in different ways.

The only thing that remains in this overview is to present the derivation of the equations by Levi-Civita and to give a critical review of it, as the historical course is determined and priority lies with Meshchersky.

In the first place, from a purely methodological point of view, Levi-Civita uses consistently the vector method and no wonder, considering the time in which he was dealing with this problem.

Secondly, Meshchersky had in mind, first of all, the separation of a mass from the primary body (rise of aerostat by throwing away the ballast), which is similar to rocket dynamics, whereas Levi-Civita thought in the first place, of the attachment of parts (particles). He had in mind the cosmic meteoric dust and the increase of the mass of planets, i.e., an astronomical problem, though also he considered the effects of the radiation of bodies.

It is very interesting, however, that Levi-Civita in his explanations also starts from the contact hypothesis and the non-elastic impact!

Levi-Civita used the following assumptions: If the mass of the primary body is $\underline{m}$ and the mass of the attached part ξ (Levi-Civita's designations) and if their absolute velocities in relation to some fixed coordinate axes are $\vec{\underline{v}}$ and $\vec{\underline{w}}$, the velocity of the center of mass $\underline{m}$ and ξ, immediately before the impact, is then

$$\frac{m\vec{v} + \xi\vec{w}}{\underline{m} + \xi} .$$

Consequently, the change $\Delta\underline{v}$ of the velocity of the primary body after the impact of mass ξ amounts to

$$\Delta\vec{\underline{v}} = \frac{m\vec{v} + \xi\vec{w}}{\underline{m} + \xi} - \vec{\underline{v}} = \frac{\xi(\vec{w} - \vec{v})}{\underline{m} + \xi}, \quad (6)$$

which, after neglecting the mass ξ of the attached particle in relation to the mass $\underline{m}$ of the primary body, gives

$$\underline{m}\Delta\vec{\underline{v}} = \xi(\vec{\underline{w}} - \vec{\underline{v}}) . \quad (7)$$

This relation, when we introducing $\xi = \Delta\underline{m}$ and instead of the designation $\vec{\underline{w}}$ write $\vec{\underline{u}}$, is reduced after dividing by $\Delta\underline{t}$ and the transition to the limit, to the relation (2). However, *Levi-Civita had not in mind the derivation of the expression for the reactive force*, but wished to derive the relation (4), which sometimes is called *Levi-Civita's equation* in the literature.

He therefore assumed that the attached mass $\Delta\underline{m}$ does not move in relation to the fixed system, that is, he assumed that it can be conceived that the contemplated fixed system is connected with the cosmic dust and then $\vec{\underline{w}} = 0$. He justified this by a condition that Oppolzer laid down in 1884, in *Astr. Nachrichten* for the planetary system. Then, from (7), after the division by $\Delta\underline{t}$ and the transition to the limit, one obtains for the force of the part d"$\vec{v}$/dt (Levi-Civita's designation) of the acceleration of the body, resulting from the reaction of the attachment of the mass $\Delta\underline{m}$, the expression

$$\underline{m}\frac{d''\vec{v}}{dt} = -\vec{\underline{v}}\frac{dm}{dt} . \quad (8)$$

If the external force $\vec{F}$ acts, too, on the contemplated body (mass m), the acceleration d'v/dt (again Levi-Civita's designation) resulting therefore is then determined by the expression

$$m \frac{d'v}{dt} = \vec{F}. \tag{9}$$

By adding the equations (8) and (9) and considering the principle of the independence of the action of forces, one obtains from here

$$m \frac{d\vec{v}}{dt} = \vec{F} - \vec{v} \frac{dm}{dt}, \tag{10}$$

where $d\vec{v}$ is the total change of the velocity of the body, caused by the external force and the reactive force, i.e., $d''\vec{v} + d'\vec{v} = d\vec{v}$.

The previous relation can immediately be written in the form (4), which was to be demonstrated.

Different approaches to the problem of motion of the body (point) of variable mass (separation, attaching), different models (serostat and cosmic dust), and furthermore different aims which have been observed (search for the equations of motion in general or only of that one where the attached, resp. separated, system does not move in relation to the fixed system) show that both investigators, Meshchersky and Levi-Civita had come to their results independently of each other. In addition, it ought to be emphasized that equation (4) is by Meshchersky a consequence of general equation (3), whereas equation (4) is the main object by Levi-Civita. With Levi-Civita the reactive force is not emphasized at all.

Briefly, also the origin of the second basic equation of rocket dynamics shall be mentioned, that is, the *formula for determining the magnitude of velocity v of the motion of the rocket in consequence of reactive propulsion.*

From the relation (2) it can be easily written in the scalar form, $m\,dv = -c\,dm$, where $dm > 0$, and dv is the change of velocity in consequence of the reactive propulsion. By means of integration from an initial moment (e.g., from the start of the rocket) to an arbitrary later moment in time and based on the assumption that c is invariable in the course of time, is derived the formula

$$v = c \ln \frac{m_0}{m} \tag{11}$$

where m_0 is the mass of the body (rocket on the start) at the beginning of the motion ($v_0 = 0$), and m is its mass at a later moment.

This is the *equation of Tsiolkovsky*, as it was established by the Russian scientist Konstantin Eduardovich Tsiolkovsky (1857-1935) and published in this work *Exploration of Cosmic Space by Reaction Apparatuses* (in Russian), 1903.

Consequently, both basic equations of rocket dynamics were discovered by Russian scientists at the end of the last century and in the beginning of this century.