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LETTERS AND COMMENTS

Comment on 'An analysis of Newton's projectile diagram'

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Abstract. Evidence is presented which corrects the conclusions of a paper in this journal (Topper D and Vincent D E 1999 *Eur. J. Phys* 20 59–66) that Newton lacked a full understanding of the motion of a projectile near the surface of the Earth.

In a recent paper [1] on Newton's famous diagram of projectile motion which appears in his book *On the System of the World*, Topper and Vincent point out an ambiguity in the text, and also in a related passage† in the *Principia*, Def. 5. They then explain this ambiguity by suggesting 'that Newton never did make a rigorous calculation of the bounds of projectiles striking the Earth', that 'instead . . . he relied on his intuition', and that consequently 'he may not have realized, even to the day of his death, that a projectile fired horizontally from the north pole can never land on the Earth past the south pole'. This conclusion, however, is untenable as will be demonstrated here from evidence in Newton's correspondence and in selected propositions in the *Principia*.

Indeed, Newton's earliest known calculations of orbital dynamics deal with projectile motion under the action of a central force [5]. In a letter written on 13 December 1679 to Robert Hooke, Newton already demonstrated that he understood quite well the geometrical shape of the orbits [6] which is, of course, relevant for the motion of a projectile near the surface of the Earth, because the gravitational force is directed towards its centre. In his celebrated Moon test in the 1660s Newton assumed that the radial dependence of gravity varies inversely with the square of the distance, which he deduced by applying Kepler's third law for planetary motion to the dynamics of circular motion (Newton, and independently Huygens, had shown that the radial acceleration or force is proportional to the square of the velocity and inversely proportional to the radius of the circular orbit). He was aware, however, that near the Earth's surface this inverse square dependence was an unlikely extrapolation, but by 1685 he was able to prove the validity of his assumption. As he remarked in the *Principia*, Book 3, Prop. 8,

> . . . I was yet in doubt whether that proportion inversely as the square of the distance did accurately hold . . . near the surface of the planet . . . But by the help of Prop. 75 and 76, Book 1 and their Corollaries, I was at last satisfied of the truth of the Proposition . . .

In his letter to Hooke, Newton pointed out that if the magnitude of the central force is constant or if it increases as the radial distance towards the centre decreases, then a projectile launched perpendicularly to the

† In a subsequent communication [2], Topper and Vincent admit that the ambiguous clause in Def. 5, quoted in their paper, appeared only in the second edition of the *Principia*, and that in the third edition Newton deleted the 'phrases pertaining to falling'; see [3]. Surprisingly, the authors maintain that 'this does not change the meaning of the key clause', and that 'the ambiguity in the projectile passage was never corrected'. To quote Koyré and Cohen, 'But in a scientific book, the earlier versions are in general merely stages towards the ultimate conception of the truths of nature – at least to the degree attained by the author in his final revision'. For the proper English translation of Def. 5 as it appears in the third edition of the *Principia*, see [4].

radial direction (with an initial velocity less than that required for circular motion) would move on an orbit in which the radial distance decreased monotonically until it reached a minimum value. At this minimum radius the velocity is again perpendicular to the radius, and in the subsequent segment of the orbit the procedure is reversed and the radius increases until it reaches its initial value. He obtained these insights by extending the dynamics of circular motion to a more general orbit [6].

At the minimum radius, the polar angle θ relative to the initial radial direction depends on the nature of the central force. In a diagram accompanying the text, Newton illustrated an approximate orbit when the magnitude of the central force is a constant, and a careful analysis of this diagram shows that he had found $\theta \approx 107°$ in this case [6]. He also pointed out that 'if it be supposed [that gravity is] greater nearer the centre' then θ increases to 180° or beyond. While in his letter Newton did not mention the nature of the force which corresponds to the special case that $\theta = 180°$, in 1684 he revealed, in a scholium apparently intended for inclusion in the *Principia* [7], that this force is an inverse square force. This scholium essentially reproduces the contents of Newton's letter to Hooke [6], but Newton cancelled it when he later obtained an expression for the general case of a central force $1/r^n$,

$$\theta = 180°/\sqrt{3-n} \qquad (1)$$

where $n < 3$. This remarkable result appears in Prop. 45 of the first edition of the *Principia*, and shows that by 1687 Newton had gained a full understanding of projectile motion under the action of a general central force and, moreover, that he had even obtained an analytic solution for orbits of small eccentricity.

Newton's expression in equation (1), when applied to a projectile launched horizontally from the top of a mountain on the Earth, gives the maximum angle θ that it can move around the Earth before hitting the surface (here the minimum radius corresponds to the radius of the Earth). In 1679 Newton was uncertain about the radial dependence of the gravitational force near the surface of the Earth, and it is reasonable to suppose that for this reason he considered the orbital problem for a general class of central forces. For example, if gravity is assumed to be a constant as it was generally believed at the time, then $n = 0$ and $\theta \approx 103.9°$. After 1685, however, Newton had shown that gravity varies inversely as the square of the distance, i.e. $n = 2$, even near the surface of the Earth. This implies that the maximum angle for the fall is $\theta = 180°$, as expected for an elliptic orbit. For this orbit a projectile launched at the north pole lands on the south pole, and any increase of velocity puts it into an elliptic orbit that returns it to the top of the mountain without any possibility of landing on the Earth past the south pole. Hence, contrary to the conclusions of Topper and Vincent [1, 2], Newton, instead of depending on his intuition, carried out rigorous calculations to obtain the path of a projectile near the surface of the Earth, and already nine years before the publication of the *Principia* he had obtained approximate quantitative results for its orbit in the case of general central forces.

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